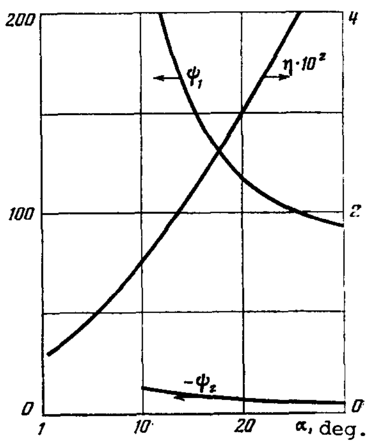
$$\psi_1 \approx \frac{1}{N_4} \ln \frac{1+q+r}{r}, \quad \psi_2 \approx \frac{1}{N_1} \ln \frac{1+q+r}{q}, \quad q = \frac{b^2 B_1^2}{B}, \; r = \frac{d^2 B_2^2}{B}$$

and we obtain the following estimates for t_1 and t_2:

$$\psi_1 (1+q+r)^{-1/2} < t_1 < 2\psi_1, \quad -\psi_2 < t_2 < -2\psi_2 \tag{3}$$

The figure shows the dependence of the roots ψ_1 and ψ_2 on α. For a given value of α the value of ψ_1 is greater than that of $|\psi_2|$ in all cases. The estimates (3) show that for a given value of α t_1 is greater than t_2.



Thus we can conclude that the direction of rotation changes much more rapidly if the body rotates in the positive direction, and this agrees with the results of numerical experiment /2/ where $\alpha = 30°$, $|\omega| = 1$ and the interval of integration was 4 min. The first experiment, where $\alpha = 5°$, $|\omega| = 5$ and the interval of integration was 30 sec., confirms this. The authors discovered the change in the direction of rotation only in the case when the initial angular velocity was positive, because the interval of integration shorter than t_1. Had they used a longer interval of integration, e.g. 2 min, they would have noticed that the change in direction occurs for either direction of the initial angular velocity.

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THE SUFFICIENT CONDITIONS FOR THE EXISTENCE OF ASYMPTOTICALLY PENDULUM-LIKE MOTIONS OF A HEAVY RIGID BODY WITH A FIXED POINT*

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The present paper continues the study of the asymptotically pendulum-like motions (APM) of a heavy rigid body begun in /1/, where a specific mass distribution is not assumed here a priori. The first Lyapunov method /2/ is used to obtain new sufficient conditions for the existence of APM, which cannot be described by the well-known particular solutions of the Euler-Poisson equations.

1. The equations of the first approximation. We shall attach to the body a special coordinate system /3/ and assume that the centre of mass lies in the principal plane of the ellipsoid of inertia constructed for the fixed point. Then the equations of motion will have the form /3/

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$$x^{\cdot} = (a_2 - a_1)\, yz - b_1 xz,\quad y^{\cdot} = (a - a_2)\, xz + b_1 zy - \Gamma\nu_2 \tag{1.1}$$
$$z^{\cdot} = (a_1 - a)\, xy + b_1 (x^2 - y^2) + \Gamma\nu_1$$
$$\nu^{\cdot} = a_2 z\nu_1 - (a_1 y + b_1 x)\,\nu_2,\quad \nu_1^{\cdot} = (ax + b_1 y)\,\nu_2 - a_2 z\nu$$
$$\nu_2^{\cdot} = (a_1 y + b_1 x)\,\nu - (ax + b_1 y)\,\nu_1$$

and will admit of the first integrals

$$ax^2 + a_1 y^2 + a_2 z^2 + 2b_1 xy - 2\Gamma\nu = 2\Gamma h \tag{1.2}$$
$$x\nu + y\nu_1 + z\nu_2 = m,\quad \nu^2 + \nu_1^2 + \nu_2^2 = 1$$

A particular solution of (1.1) describing the rotation of the body about the horizontal axis (the motion of a physical pendulum) is /4, 5/:

$$x = x^* = 0,\quad y = y^* = 0,\quad z = z^* = a_2^{-1}\varphi^{\cdot} \tag{1.3}$$
$$\nu = \nu^* = \cos\varphi,\quad \nu_1 = \nu_1^* = -\sin\varphi,\quad \nu_2 = \nu_2^* = 0;$$
$$\varphi^{\cdot} = [2a_2\Gamma (h^* + \cos\varphi)]^{1/2}$$

Here h^* is the normalized constant of the energy integral in the motion in question.

The equations of the first approximation for the solution (1.3) were studied earlier in /5/ form the point of view of integration in closed form. The Lyapunov stability of the solution in question was studied in /4/ under the condition that the amplitude of the deviation of the centre of mass of the body from its position of stable equilibrium was small, and the APM of the Hess-Appel'rot gyroscope were constructed in /1/. Here we shall study the conditions for the existence of the non-zero eigenvalues of the first-approximation system for the motion of a physical pendulum, under the constraint $h^* > 1$.

Let us write $x = x^* + x_1$, $y = y^* + x_2$, $\nu_2 = \nu_2^* + x_3$, $z = z^* + y_1$, $\nu = \nu^* + y_2$, $\nu_1 = \nu_1^* + y_3$ and adopt the function of time $\varphi(t)$ monotically increasing without limit as $t \to \infty$, as the new independent variable, and denote differentiation with respect to φ by a prime.

The first-approximation system for the solution (1.3) has two closed subsystems. The first subsystem

$$x_1' = -a_2^{-1} b_1 x_1 + (1 - a_2^{-1} a_1)\, x_2 \tag{1.4}$$
$$x_2' = (a_2^{-1} a - 1)\, x_1 + a_2^{-1} b_1 x_2 - (a_2 z^*)^{-1} \Gamma x_3$$
$$x_3' = (a_2 z^*)^{-1} [(b_1 \nu^* - a\nu_1^*)\, x_1 + (a_1 \nu^* - b_1 \nu_1^*)\, x_2]$$

admits of a single integral

$$\nu^* x_1 + \nu_1^* x_2 + z^* x_3 = I_1 \tag{1.5}$$

and the second system

$$y_1' = (a_2 z^*)^{-1}\, \Gamma y_3,\quad y_2' = (z^*)^{-1} \nu_1^* y_1 + y_3 \tag{1.6}$$
$$y_3' = -(z^*)^{-1} \nu^* y_1 - y_2$$

admits of two integrals

$$a_2 z^* y_1 - \Gamma y_2 = I_2,\quad \nu^* y_2 + \nu_1^* y_3 = I_3 \tag{1.7}$$

The integrals (1.5), (1.7) are obtained by linearizing the first integrals (1.2) in the neighbourhood of the solution (1.3).

All the eigenvalues of system (1.6) are zeros /1/. Let us now investigate the eigenvalues of (1.4).

Let A, B, C be the principal moments of inertia of the body and e_1, e_2, e_3 $(e_3 = 0)$ the direction cosines of the ray connecting the fixed point with the centre of mass in the principal axes; the centre of mass lies in the principal plane of the ellipsoid of inertia orthogonal to the axis, the moment of inertia relative to which is equal to C. We write

$$\alpha = AC^{-1},\quad \beta = BC^{-1}\qquad \lambda = a_2^{-1} (a_2 - a_1) = \alpha^{-1}\beta^{-1} [\alpha (\beta - 1)\, e_1^2 + \beta (\alpha - 1)\, e_2^2]$$
$$\mu = a_2^{-2} [(a_2 - a_1)(a - a_2) + b_1^2] = \alpha^{-1}\beta^{-1} (1 - \alpha)(\beta - 1)$$
$$\varkappa = a_2^{-1} b_1 = \alpha^{-1}\beta^{-1} (\beta - \alpha)\, e_1 e_2$$
$$\mu_1 = 2\mu + \lambda = \alpha^{-1}\beta^{-1} [(\beta - 1)(2 - \alpha)\, e_1^2 + (\alpha - 1)(2 - \beta)\, e_2^2]$$

The case of $\lambda = 0$ was dealt with in /1/; we shall therefore assume from now on that $\lambda \neq 0$.

Using the integral (1.5), we write the variables x_2 and x_3 in terms of x_1 thus:

$$x_2 = \lambda^{-1} (x_1' + \varkappa x_1),\quad x_3 = (z^*)^{-1} (I_1 - x_1 \cos\varphi + x_2 \sin\varphi) \tag{1.8}$$

The variable x_1 satisfies the equation /5/

$$2\,(h^* + \cos\varphi)\, x_1'' + x_1' \sin\varphi + (-\mu_1 \cos\varphi + \varkappa \sin\varphi - 2\mu h^*)\, x_1 = -\lambda I_1 \tag{1.9}$$

The relations (1.8) show that the set of eigenvalues (EV) of system (1.4) consists of the EV of the homogeneous equation corresponding to (1.9), and zero.

Let us write in (1.9) $I_1 = 0$ and carry out the periodic change of variable $x_1 = (h^* + \cos\varphi)^{1/4} w$. The change variable does not affect the EV, and transforms (1.9), when $I_1 = 0$, into the Hill equation

$$w'' - [4\,(h^* + \cos\varphi)]^{-2}\, \Phi(\varphi)\, w = 0 \tag{1.10}$$
$$\Phi(\varphi) = 16\mu h^{*2} + [4\,(8\mu + 2\lambda + 1) \cos\varphi - 8\varkappa \sin\varphi] h^* + (8\mu + 4\lambda - 1/2) \cos 2\varphi - 4\varkappa \sin 2\varphi + 8\mu + 4\lambda + 9/2$$

Thus the first-approximation system for the motion of a physical pendulum (1.3) has, when $h^*>1$, four zero and two non-zero EV of different sign, if and only if the Hill Eq.(1.10) has a pair of non-zero EV.

2. Asymptotically pendulum-like motions. In order to study Eq.(1.10) we shall use the simplest sufficient condition for the existence of a non-zero eigenvalue /2/

$$\Phi(\varphi) \geqslant 0 \quad \text{for any } \varphi \tag{2.1}$$

We note that Φ cannot vanish identically.

We can obtain a set of sufficient conditions for the inequality (2.1) to hold, using the following case. We will regard Φ as a polynomial in h^* of degree not higher than the second, and φ as a parameter.

If $\mu>0$, which happens when either $\beta>1>\alpha$ or $\alpha>1>\beta$, then the inequality (2.1) holds for fairly large h^*. We can, for example, assume that

$$h^* > \max\{1, h_0^*\} \tag{2.2}$$

$$h_0^* = (16\mu)^{-1}\left\{4\left[(4\mu+2\mu_1+1)^2+(2\varkappa)^2\right]^{1/2}+\left[(4\mu_1-{}^1/_2)^2+(4\varkappa)^2\right]^{1/2}+\left|4\mu_1+\frac{9}{2}\right|\right\}$$

The condition $\mu>0$ means that the centre of mass of the body lies in the plane orthogonal to the middle axis of the ellipsoid of inertia.

We can formulate the following problem: it is required to find a mass distribution such that the inequality (2.1) holds for any $h^*>1$.

First we consider the case when $\mu=0$. In this case the problem has a solution if and only if $\lambda=-{}^1/_2$, $\varkappa=0$. The conditions for the distribution of mass within the body have the form

$$e_1=1,\ e_2=e_3=0,\ \alpha=1,\ \beta={}^2/_3 \tag{2.3}$$

When $\mu>0$, the solution of the above problem becomes slightly more complicated, since Φ is a quadratic function of h^*. In this case it is sufficient to require that the discriminant $\Delta(\varphi)$ of the quadratic trinomial $\Phi(h^*)$ be positive for any φ:

$$\Delta(\varphi)=(2\lambda+1)^2-20\left(\mu-\frac{\varkappa^2}{5}\right)+\left[(2\lambda+1)^2+20\left(\mu-\frac{\varkappa^2}{5}\right)\right]\cos 2\varphi-4\varkappa(2\lambda+1)\sin 2\varphi\leqslant 0 \tag{2.4}$$

Calculations show that the inequality (2.4) holds if and only if $\lambda=-{}^1/_2$, $\varkappa^2\leqslant 5\mu$. From this we obtain, using the triangular inequalities for the moments of inertia A, B, C, the following restrictions for the distribution of mass:

$$e_1^2=\frac{\beta(2-3\alpha)}{2(\beta-\alpha)},\quad e_2^2=\frac{\alpha(3\beta-2)}{2(\beta-\alpha)},\quad e_3=0 \tag{2.5}$$

and either

$$\alpha={}^2/_3,\ \beta\in\,]1;\,{}^5/_3[,$$

or

$$\alpha\in\left]\frac{17-\sqrt{201}}{22};\,\frac{2}{3}\right[,\ \beta\in\left[\frac{16-14\alpha}{14-11\alpha};\,1+\alpha\right[$$

The fact that the centre of mass of the body need not lie on the principal axis, is of interest.

If $\mu>0$, then inequality (2.1) will hold for all $h^*>1$, and in the case when the largest root of the equation $\Phi(h^*)=0$ is less than unity for all values of φ for which $\Delta(\varphi)\geqslant 0$. In order for this situation to occur, it is sufficient to require that $\varkappa=0$ and $-4\mu-{}^1/_2<\lambda<-{}^1/_2$. Let us write the corresponding conditions for the parameters of the problem

$$e_1=1,\ e_2=e_3=0 \tag{2.6}$$

$$\alpha\in\left]\frac{9-\sqrt{41}}{10};\,\frac{2}{3}\right[,\quad \beta\in\left]\frac{8(1-\alpha)}{6-5\alpha};\,1+\alpha\right[$$

Applying the first Lyapunov method /2/ and the method of investigating the orbital stability with the help of the linear approximation /6/, we shall formulate the results obtained in terms of the following assertion.

Let h^* satisfy (2.2) and either $(1-\alpha)(\beta-1)>0$ or $h^*>1$, and let the parameters $\alpha, \beta, e_1, e_2, e_3$ be connected by any one of the following constraints (2.3), (2.5), (2.6). Then: 1) the equations in variations (1.4), (1.6) have one positive and one negative eigenvalue; 2) a one-parameter family of motions of the body exists, which tend, as the time increases without limit, asymptotically to an orbitally unstable rotation (1.3) about the horizontal axis.

We note that the asymptotically pendulum-like motions of gyroscopes with the mass distribution shown above cannot be described by the particular solutions of the Euler-Poisson equations known at the present time.

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COMPRESSION OF A MULTILAYERED MEDIUM UNDER THE ACTION OF A VARIABLE EXTERNAL PRESSURE*

M.YU. IVANOV, V.K. KOROBOV, V.M. NIKOLAYENKO and K.P. STANYUKOVICH

The solutions of equations describing the system of waves that arise when a rigid body is compressed by means of a pressure that varies with time, are obtained in the acoustic approximation. The case when the compressed medium consists of two layers of different initial density is considered. The solutions obtained can be generalized to the case of the compression of a multilayer medium.

1. Let us consider the wave motions in a continuum under the action of a variable pressure $p = p(t)$ applied at the boundary of the medium. The equation of state for the rigid continuum is usually given in the form $p - p_0 = A(\rho^n - \rho_0^n) + B\rho T$, with the corresponding equation of the adiabatic curve $p - p_0 = A(\rho^n - \rho_0^n) + B_1(\rho^m \exp[(s - s_0)/c_v] - \rho_0^m)$. We shall replace the latter by the simpler equation of an adiabatic curve

$$p = A(s)\rho^\gamma - B \tag{1.1}$$

When the deformations of an elastic solid are small, Hooke's law $\sigma = k\varepsilon$ holds, where k is the bulk modulus ε is the deformation, $\varepsilon = (v - v_0)/v_0 = \rho_0/\rho - 1$, σ is the stress and $-\sigma = p$. We shall require that Eq.(1.1), in the linear approximation with respect to the deformation, shall be the same as Hooke's law $-\sigma = A\rho_0^\gamma - B - \gamma A\rho_0^\gamma\varepsilon$. This yields $A\rho_0^\gamma = B$, $\gamma A\rho_0^\gamma = k$. We note that the velocity of sound $c_0^2 = \gamma A\rho_0^{\gamma-1} = k/\rho$. Knowing the bulk modulus or the velocity of sound, and specifying the quantity γ, we can easily find the constants A and B for use in approximating (1.1).

We shall use the equations describing the propagation of the wave system in Lagrangian form, transforming them for convenience to the independent variables h, p:

$$\begin{gathered} u_p = r^N t_h, \quad x_h - v = x_p t_h / t_p \\ r_p = u t_p, \quad s_p = 0 \\ \left(h = \int_0^r \rho r^N\, dr, \quad x = r^{N+1}/(N+1)\right) \end{gathered} \tag{1.2}$$

Here r is the Eulerian coordinate, h is the Lagrangian mass coordinate, u is the velocity and s is the entropy; the subscript denotes the derivative with respect to the corresponding variable; $N = 0, 1, 2$ for plane, cylindrical and spherical symmetry respectively.

Using the first and third equation of (1.2), we transform the second equation of (1.2) as follows:

$$x_h = v + u u_p \tag{1.3}$$

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